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The axial anomaly in the light-cone gauge

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Abstract. We evaluate the axial anomaly using the light-cone gauge and light-cone coordinates. In addition to the generally accepted result we find extra non-Lorentz-covariant contributions.

1. Introduction

The problem of the axial anomaly has its origins with a paper by Steinberger in 1949. Since then our understanding of the phenomenon has been much improved (Schwinger 1951, Adler 1969, 1970, Bell and Jackiw 1969, Bardeen 1969, Adler and Bardeen 1969) and, in particular, recent interest has concerned the question as to whether the Adler-Bardeen theorem (1969) holds in supersymmetric theories (Jones and Leveille 1982a, b, Grisaru and West 1983, Breitenlohner *et al* 1984). Another, different, area of current interest is the formulation of supersymmetric field theories and string theories in the light-cone gauge (Mandelstam 1983, Green and Schwartz 1981, Alvarez-Gaume and Witten 1983). In the case of supersymmetric field theories the light-cone gauge is central to Mandelstam's proof of the finiteness of the $N=4$ supersymmetric Yang-Mills theory (Capper *et al* 1984) as well as the off-shell formulation of $N=8$ supergravity. In view of the importance of the light-cone gauge it is desirable to investigate the validity of the formulation in a more familiar context; in this paper we calculate the anomaly for quantum electrodynamics.

In order to handle the divergent integrals that occur in any quantum field theory we need to introduce a regularisation scheme. Here we employ dimensional regularisation ('t Hooft and Veltman 1972, Ashmore 1972, Bollini and Giambiagi 1972, Leibbrandt 1975) since it respects most symmetries and, perhaps more significantly, enables us to evaluate what would otherwise be very difficult integrals. As is well known (Akyeampong and Delbourgo 1973, 1974, Breitenlohner and Maison 1977, Bonneau 1981), however, there exists a substantial problem in applying dimensional regularisation to theories which involve the γ_5 matrix. The two assumptions of cyclicity of the trace and a totally anticommuting γ_5 enable one to show in an n-dimensional space that

$$(n-4)\,\mathrm{Tr}\,\gamma_5\gamma_\alpha\gamma_\beta\gamma_\rho\gamma_\sigma=0. \tag{1.1}$$

This result demonstrates that there is no *analytic* continuation of γ_5 with the above properties. Moreover, it is precisely this inconsistency between analyticity, an anticommuting γ_5 and cyclicity of the trace which gives rise to the axial anomaly; otherwise it would be possible to show that the anomaly vanishes (Adler 1970). There have been

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various attempts to define the γ_5 matrix in the context of dimensional regularisation (Akyeampong and Delbourgo 1973, 1974, Breitenlohner and Maison 1977, Bonneau 1981). However, as shown in Jones and Leveille (1982a, b) and Capper (1979), one can get consistent results for the axial anomaly without assuming any properties of the γ_5. One simply avoids moving the γ_5 matrix through any of the other γ-matrices until after the integrals have been evaluated. It can then be seen that the anomaly is finite and once this stage has been reached an ordinary four-dimensional γ_5 can be employed. The generally accepted result for the anomaly then follows. In this paper we make use of the same approach but in the context of the light-cone gauge.

2. The axial anomaly in the light-cone gauge

We start from the usual Lagrangian for quantum electrodynamics†:

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\psi}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi. \tag{2.1}$$

Light-cone coordinates are introduced by defining, for an arbitrary vector X^μ,

$$X^\pm = (1/\sqrt{2})(X^0 \pm X^3). \tag{2.2}$$

Spinors $\psi_\pm$ are defined by

$$\psi_\pm = \tfrac{1}{2}\gamma_\pm\gamma_\mp\psi \tag{2.3}$$

where

$$\gamma_\pm = (1/\sqrt{2})(\gamma_0 \pm \gamma_3). \tag{2.4}$$

The light-cone gauge consists of imposing the condition

$$A^+ = 0. \tag{2.5}$$

If we now choose X^+ to be our 'time' coordinate then A^- and ψ_- can be eliminated via the equations of motion‡

$$A^- = \frac{1}{\partial^+}\left(\partial^i A^i + \frac{\mathrm{i}e\sqrt{2}}{\partial^+}(\psi_+^*\psi_+)\right), \tag{2.6}$$

$$\psi_- = -\frac{1}{2\partial^+}\gamma_-(\hat{\not{\partial}}\psi_+ + \mathrm{i}e\hat{\not{A}}\psi_+). \tag{2.7}$$

We can thus express the Lagrangian as

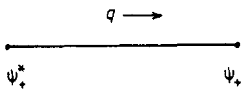
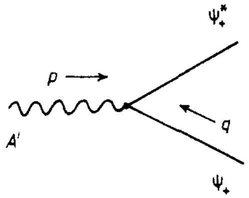
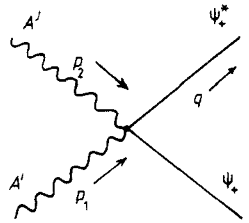
$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}A^i\partial^2 A^i + \frac{1}{\sqrt{2}}\psi_+^*\frac{\partial^2}{\partial^+}\psi_+ - \mathrm{i}e\sqrt{2}\,\partial^i A^i\frac{1}{\partial^+}(\psi_+^*\psi_+) + \frac{\mathrm{i}e}{\sqrt{2}}\psi_+^*\frac{\hat{\not{\partial}}}{\partial^+}(\hat{\not{A}}\psi_+) \\ & + \frac{\mathrm{i}e}{\sqrt{2}}\psi_+^*\frac{\hat{\not{A}}\hat{\not{\partial}}}{\partial^+}\psi_+ - \frac{e^2}{\sqrt{2}}\psi_+^*\hat{\not{A}}\frac{1}{\partial^+}(\hat{\not{A}}\psi_+) + \frac{e^2}{\partial^+}(\psi_+^*\psi_+)\frac{1}{\partial^+}(\psi_+^*\psi_+)\end{aligned} \tag{2.8}$$

which gives rise to the Feynman rules shown in table 1.

† We use a $(+---)$ metric.

‡ In order to avoid confusion we denote the Hermitian conjugate of ψ_+ by ψ_+^*. We denote the $\mu = 1, 2$ components of a vector X^μ by X^i and when using dimensional regularisation continue the i components to $(2\omega - 2)$ dimensions. Scalar products of these restricted vectors are distinguished via $\hat{X}\cdot\hat{Y} = \Sigma_i X^i Y^i$.

Table 1. Feynman rules relevant to our calculation which arise from (2.8).

Diagrammatic representation	Momentum space description
	$-\dfrac{i\sqrt{2}\,q^+}{q^2}$
	$-\dfrac{ie}{\sqrt{2}}\left(-\dfrac{2p^i}{p^2}+\dfrac{\gamma^i\hat{q}}{q^+}+\dfrac{(\hat{q}+\hat{p})\gamma^i}{(q^+ + p^+)}\right)$
	$ie^2\sqrt{2}\left(\dfrac{\gamma^i\gamma^j}{(q^+ - p_1^+)}-\dfrac{\gamma^j\gamma^i}{(q^+ - p_2^-)}\right)$

The chiral invariance of the original Lagrangian (equation (2.1)) leads us to introduce the axial current

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{2.9}$$

and our objective is to evaluate

$$\langle 0|\partial_\rho J_5^\rho A^\mu(p_1)A^\nu(p_2)|0\rangle. \tag{2.10}$$

One approach is to calculate the diagrams shown in figure 1 using the method of Capper (1979). There, using the vertex derived from $\partial_\mu J_5^\mu$ and assuming no properties of the γ_5 matrix until after the integrals had been evaluated, a finite result was obtained. It turned out to be very hard to carry out this program in the light-cone gauge, mainly due to the difficulty in performing the integrations. In fact we were able to carry this calculation through to completion only by assuming an anticommuting γ_5 matrix; even then the evaluation of the integrals involved hypergeometric functions. This technique was unsuccessful since the result for the anomaly turned out to be zero!

We now revert to the alternative approach given in Jones and Leveille (1982a, b) which in the covariant formulation consists of first defining

$$R^{\sigma\mu\nu}(p_1, p_2) = \langle 0|J_5^\sigma A^\mu(p_1)A^\nu(p_2)|0\rangle. \tag{2.11}$$

On the basis of symmetry and Lorentz covariance we can also write

$$\langle 0|\partial_\sigma J_5^\sigma A^\mu(p_1)A^\nu(p_2)|0\rangle = A\mathscr{E}^{\mu\nu\rho\sigma}p_{1_\rho}p_{2_\sigma} \tag{2.12}$$

where the coefficient A is the anomaly we wish to calculate. $R^{\sigma\mu\nu}$ and A are related by

$$i(p_1+p_2)_\sigma R^{\sigma\mu\nu}(p_1, p_2) = A\mathscr{E}^{\mu\nu\rho\sigma}p_{1_\rho}p_{2_\rho} \tag{2.13}$$

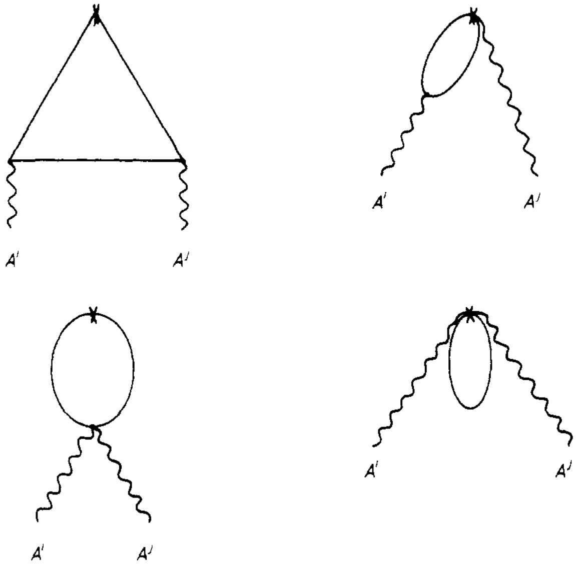


Figure 1. Fermion loop contributions to $\langle 0|\partial_\rho J_5^\rho A^i(p_1)A^j(p_2)|0\rangle$. The upper vertex results from the $\partial_\rho J_5^\rho$ insertion.

and the technique used in Jones and Leveille (1982a, b) is to differentiate with respect to p_2^σ and then put

$$p_1 = p = -p_2 \tag{2.14}$$

thus obtaining

$$\mathrm{i}R^{\sigma\mu\nu}(p, -p) = A\mathscr{E}^{\mu\nu\rho\sigma}p_\rho. \tag{2.15}$$

To find A we only need to consider $R^{\sigma\mu\nu}(p, -p)$ which is far easier than $R^{\sigma\mu\nu}(p_1, p_2)$.

This approach can be applied to the light-cone coordinate calculation and we begin by rewriting (2.13) as

$$\begin{aligned}\mathrm{i}(p_1^- + p_2^-)R^{+ij}(p_1, p_2) + \mathrm{i}(p_1^+ + p_2^+)R^{-ij}(p_1, p_2) - \mathrm{i}(p_1^m + p_2^m)R^{mij}(p_1, p_2)\\ = A[\mathscr{E}^{ij+-}p_1^- p_2^+ + \mathscr{E}^{ij-+}p_1^+ p_2^-].\end{aligned} \tag{2.16}$$

The major difference, however, is that when we differentiate (2.16) with respect to p_2^σ it is possible to obtain terms of the form $(p_1^+ + p_2^+)^{-1}$ from the R^{-ij} factors. Taking this into account and eliminating those contributions that are clearly zero we obtain

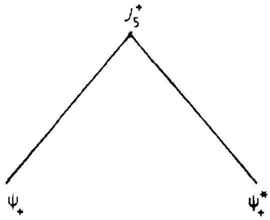
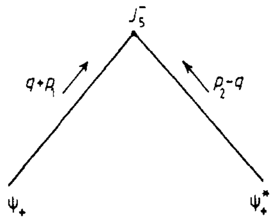
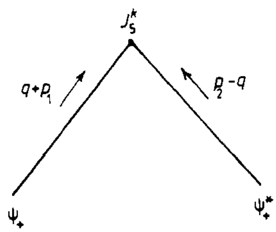
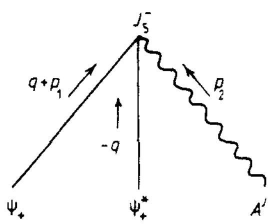
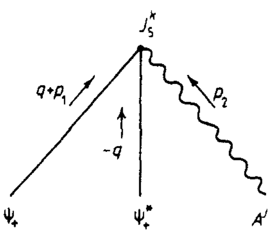
$$\mathrm{i}(p_1^+ + p_2^+)\frac{\mathrm{d}}{\mathrm{d}p_2^-}[R^{-ij}(p_1, p_2)]_{p_1=-p_2=p} + \mathrm{i}R^{+ij}(p, -p) = A\mathscr{E}^{ij-+}p^+, \tag{2.17}$$

$$\mathrm{i}(p_1^+ + p_2^+)\frac{\mathrm{d}}{\mathrm{d}p_2^+}[R^{-ij}(p_1, p_2)]_{p_1=-p_2=p} + \mathrm{i}R^{-ij}(p, -p) = A\mathscr{E}^{ij+-}p^-, \tag{2.18}$$

$$\mathrm{i}(p_1^+ + p_2^+)\frac{\mathrm{d}}{\mathrm{d}p_2^k}[R^{-ij}(p_1, p_2)]_{p_1=-p_2=p} + \mathrm{i}R^{kij}(p, -p) = 0. \tag{2.19}$$

In order to evaluate the left-hand side of equations (2.17)-(2.19) we need the additional Feynman rules resulting from the J_μ^5 insertion; these are given in table 2. The diagrams

Table 2. Feynman rules (relevant to our calculation) resulting from a J_5^μ insertion.

Diagrammatic representation	Momentum space description
	$\frac{i}{\sqrt{2}}\gamma_5\gamma_+\gamma_-$
	$\frac{-i}{\sqrt{2}}\frac{(\hat{q}-\hat{p}_2)}{(q^+-p_2^+)}\gamma_5\gamma_-\gamma_+\frac{(\hat{q}+\hat{p}_1)}{(q^++p_1^+)}$
	$\frac{i}{2\sqrt{2}}\left(\frac{(\hat{q}-\hat{p}_2)}{(q^+-p_2^+)}\gamma^k\gamma_5\gamma_+\gamma_-+\gamma^k\gamma_5\gamma_-\gamma_+\frac{(\hat{q}+\hat{p}_1)}{(q^++p_1^+)}\right)$
	$\frac{-ie}{2\sqrt{2}}\left(\frac{\gamma^j\gamma_5\gamma_-\gamma_+(\hat{q}+\hat{p}_1)}{(q^+-p_2^+)(q^++p_1^+)}+\frac{\hat{q}\gamma_5\gamma_-\gamma_+\gamma^j}{q^+(q^++p_1^++p_2^+)}\right)$
	$\frac{ie}{2\sqrt{2}}\left(\frac{\gamma^j\gamma_5\gamma_-\gamma_+\gamma^k}{(q^++p_1^++p_2^+)}+\frac{\gamma^k\gamma^j\gamma_5\gamma_+\gamma_-}{(q^+-p_2^+)}\right)$

to be evaluated are the same as in figure 1 except that the upper vertex comes from a J_μ^5 rather than $\partial^\mu J_\mu^5$ insertion. A considerable number of integrals are required and in order to facilitate the task of checking our results some of these are given in the appendix. The algebraic manipulation program SCHOONSCHIP (Strubbe 1974) was used to aid the calculation and on evaluating the left-hand sides of (2.17)-(2.19) we obtained:

(+) term:

$$-(e^2/2\pi^2)\mathscr{E}^{ij}p^+; \tag{2.20}$$

(−) term:

$$\frac{e^2}{4\pi^2}\mathscr{E}^{ij}p^-\left(\frac{1+2p^+p^-}{p^2}\right)-\frac{ie^2}{2\pi^2p^+}(\mathscr{E}^{im}p^mp^j-\mathscr{E}^{jm}p^mp^i); \tag{2.21}$$

(k) term:

$$(e^2/2\pi^2)(\mathscr{E}^{ik}p^j-\mathscr{E}^{jk}p^i). \tag{2.22}$$

3. Discussion

From (2.20)-(2.22) we can see that the (+) term is consistent with the accepted covariant expression for the anomaly but this is not true of either the (−) or the (k) contributions. In fact the (−) term would imply an additional non-local anomaly of the form

$$\ln(p_1^2/p_2^2)(p_1^+-p_2^+)(p_1^--p_2^-)\mathscr{E}^{ij}.$$

Evidently in this calculation the breaking of Lorentz covariance is highly non-trivial and the anomaly is *not* of the form assumed in (2.16). This result may be the outcome of choosing a particular regularisation technique which appears to have moved the anomaly from the axial current to the vector current. The peculiar phenomenon of non-locality might be similar in origin to that occurring in another light-cone calculation, namely the non-local infrared divergent term in the expression for the one-loop gluon self-energy (Capper *et al* 1984).

In both of these calculations the principal value prescription was used in order to handle the singular integrals occurring in the light-cone gauge. However, use of the Mandelstam prescription removed the non-local divergent terms in the gluon self-energy. We suspect that use of this prescription in the anomaly calculation would lead to the generally accepted result. Such a calculation would be very difficult to carry out in practice.

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Appendix

In this appendix we list some of the more basic integrals used in our calculation. They were obtained by employing the principal value prescription (Capper *et al* 1984).

$$\int\frac{d^{2\omega}q}{q^2(q+p)^2[(q+p)n]^2}=\frac{2(2\omega-3)W_2}{(\omega-3)(pn)^2},$$

$$\int \frac{\mathrm{d}^{2\omega}q\, q_i}{q^2(q+p)^2[(q+p)n]^2} = \frac{(1-\omega)(3-2\omega)\, W_2 p_i}{(2-\omega)(\omega-3)(pn)^2},$$

$$\int \frac{\mathrm{d}^{2\omega}q\, q_i q_j}{q^2(q+p)^2[(q+p)n]^2} = \frac{\omega(1-\omega)\, W_2 p_i p_j}{(2-\omega)(\omega-3)(pn)^2} - \frac{W_1 \delta_{ij}}{2(2-\omega)(pn)^2},$$

$$\int \frac{\mathrm{d}^{2\omega}q\, q_i q_j q_k}{q^2(q+p)^2[(q+p)n]^2} = \frac{\omega(\omega+1)\, W_2 p_i p_j p_k}{2(2-\omega)(\omega-3)(pn)^2} - \frac{\omega W_1(\delta_{ij}p_k + \delta_{ik}p_j + \delta_{jk}p_i)}{4(1-\omega)(2-\omega)},$$

$$\int \frac{\mathrm{d}^{2\omega}q}{q^2[(q+p)^2]^2[(q+p)n]^2} = \frac{2(2-\omega)(2\omega-3)(2\omega-5)\, W_3}{(\omega-3)(\omega-4)(pn)^2},$$

$$\int \frac{\mathrm{d}^{2\omega}q}{q^2[(q+p)^2]^2(q+p)n} = \frac{2(2\omega-3)(2-\omega)\, W_3}{(\omega-3)pn},$$

$$\int \frac{\mathrm{d}^{2\omega}q}{q^2[(q+p)^2]^2 qn} = \frac{2(2\omega-3)\, W_3}{pn},$$

where

$$W_i = \frac{-i(-\mathrm{i}\pi)^{\omega}\Gamma(2-\omega)\Gamma(\omega-1)\Gamma(\omega-1)(p^2)^{\omega-i}}{\Gamma(2\omega-2)}.$$

All other integrals are derived from the above results by differentiation with respect to p_i.

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